Calculus 1
Workbook Solutions

Tangent and normal lines

krista king
MATH

TANGENT LINES

■ 1. Find the equation of the tangent line to the graph of the equation at $(1/2,\pi)$.

$$f(x) = 4 \arctan 2x$$

Solution:

The derivative of $\arctan x$ is given by

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

So the derivative is

$$f'(x) = \frac{4}{1+(2x)^2} \cdot 2$$

$$f'(x) = \frac{8}{1+4x^2}$$

Evaluating the derivative at $(1/2,\pi)$, we get

$$f'(2) = \frac{8}{1+4\left(\frac{1}{2}\right)^2} = \frac{8}{1+1} = \frac{8}{2} = 4$$

© Krista King Math

Now we can find the equation of the tangent line by plugging the slope $f'(1/2) = 4$ and the point $(1/2, \pi)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f\left(\frac{1}{2}\right) + 4\left(x - \frac{1}{2}\right)$$

$$y = \pi + 4\left(x - \frac{1}{2}\right)$$

$$y = \pi + 4x - 2$$

$$y = 4x + \pi - 2$$

■ 2. Find the equation of the tangent line to the graph of the equation at $(-1, -9)$.

$$g(x) = x^3 - 2x^2 + x - 5$$

Solution:

The derivative is

$$g'(x) = 3x^2 - 4x + 1$$

Evaluating the derivative at $(-1, -9)$, we get

$$g'(-1) = 3(-1)^2 - 4(-1) + 1$$

© Krista King Math

$$g'(-1) = 3(1) + 4(1) + 1$$

$$g'(-1) = 3 + 4 + 1$$

$$g'(-1) = 8$$

Now we can find the equation of the tangent line by plugging the slope $g'(-1) = 8$ and the point $(-1, -9)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = g(-1) + 8(x - (-1))$$

$$y = -9 + 8(x + 1)$$

$$y = -9 + 8x + 8$$

$$y = 8x - 1$$

■ 3. Find the equation of the tangent line to the graph of the equation at $(0, -4)$.

$$h(x) = -4e^{-x} + 3x$$

Solution:

The derivative is

$$h'(x) = -4(-1)e^{-x} + 3$$

© Krista King Math

$$h'(x) = 4e^{-x} + 3$$

Evaluating the derivative at $(0, -4)$, we get

$$h'(0) = 4e^{-0} + 3$$

$$h'(0) = 4(1) + 3$$

$$h'(0) = 7$$

Now we can find the equation of the tangent line by plugging the slope $h'(0) = 7$ and the point $(0, -4)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = h(0) + 7(x - 0)$$

$$y = -4 + 7(x - 0)$$

$$y = -4 + 7x$$

$$y = 7x - 4$$

■ 4. Find the equation of the tangent line to the graph of the equation at $(1,1)$.

$$f(x) = -6x^4 + 4x^3 - 3x^2 + 5x + 1$$

Solution:

© Krista King Math

The derivative is

$$f'(x) = -24x^3 + 12x^2 - 6x + 5$$

Evaluating the derivative at $(1,1)$, we get

$$f'(1) = -24(1)^3 + 12(1)^2 - 6(1) + 5$$

$$f'(1) = -24 + 12 - 6 + 5$$

$$f'(1) = -13$$

Now we can find the equation of the tangent line by plugging the slope $f'(1) = -13$ and the point $(1,1)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f(1) - 13(x - 1)$$

$$y = 1 - 13x + 13$$

$$y = -13x + 14$$

© Krista King Math

VALUE THAT MAKES TWO TANGENT LINES PARALLEL

■ 1. Find the value of a such that the tangent lines to $f(x) = 2x^3 + 2$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$f(a) = 2a^3 + 2$$

$$f'(x) = 6x^2$$

$$f'(a) = 6a^2$$

So the equation of the tangent line at $x = a$ is

$$y = f(a) + f'(a)(x - a)$$

$$y = 2a^3 + 2 + (6a^2)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$f(a + 1) = 2(a + 1)^3 + 2$$

$$f(a + 1) = 2a^3 + 6a^2 + 6a + 4$$

$$f'(x) = 6x^2$$

© Krista King Math

$$f'(a+1) = 6a^2 + 12a + 6$$

So the equation of the tangent line at $x = a + 1$ is

$$y = f(a+1) + f'(a+1)(x - (a+1))$$

$$y = 2a^3 + 6a^2 + 6a + 4 + (6a^2 + 12a + 6)(x - (a+1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$6a^2 = 6a^2 + 12a + 6$$

$$0 = 12a + 6$$

$$-12a = 6$$

$$a = -\frac{1}{2}$$

The slope of $f(x)$ at $x = -1/2$ is $3/2$, and the slope of $f(x)$ at $x = 1/2$ is $3/2$.

■ 2. Find the value of a such that the tangent lines to $g(x) = x^3 + x^2 + 7$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

© Krista King Math

$$g(a) = a^3 + a^2 + 7$$

$$g'(x) = 3x^2 + 2x$$

$$g'(a) = 3a^2 + 2a$$

So the equation of the tangent line at $x = a$ is

$$y = g(a) + g'(a)(x - a)$$

$$y = a^3 + a^2 + 7 + (3a^2 + 2a)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$g(a + 1) = (a + 1)^3 + (a + 1)^2 + 7$$

$$g(a + 1) = a^3 + 4a^2 + 4a + 10$$

$$g'(a) = 3a^2 + 2a$$

$$g'(a + 1) = 3a^2 + 8a + 5$$

So the equation of the tangent line at $x = a + 1$ is

$$y = g(a + 1) + g'(a + 1)(x - (a + 1))$$

$$y = a^3 + 4a^2 + 4a + 10 + (3a^2 + 8a + 5)(x - (a + 1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$3a^2 + 2a = 3a^2 + 8a + 5$$

$$-6a = 5$$

© Krista King Math

$$a = -\frac{5}{6}$$

The slope of $g(x)$ at $x = -5/6$ is $5/12$, and the slope of $g(x)$ at $x = 1/6$ is $5/12$.

■ 3. Find the value of a such that the tangent lines to $h(x) = \tan^{-1} x$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$h(a) = \tan^{-1} a$$

$$h'(x) = \frac{1}{1 + x^2}$$

$$h'(a) = \frac{1}{1 + a^2}$$

So the equation of the tangent line at $x = a$ is

$$y = h(a) + h'(a)(x - a)$$

$$y = \tan^{-1} a + \frac{1}{1 + a^2}(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$h(a + 1) = \tan^{-1}(a + 1)$$

© Krista King Math

$$h'(a) = \frac{1}{1 + a^2}$$

$$h'(a + 1) = \frac{1}{1 + (a + 1)^2}$$

$$h'(a + 1) = \frac{1}{a^2 + 2a + 2}$$

So the equation of the tangent line at $x = a + 1$ is

$$y = h(a + 1) + h'(a + 1)(x - (a + 1))$$

$$y = \tan^{-1}(a + 1) + \frac{1}{a^2 + 2a + 2}(x - (a + 1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$\frac{1}{1 + a^2} = \frac{1}{a^2 + 2a + 2}$$

$$1 + a^2 = a^2 + 2a + 2$$

$$-1 = 2a$$

$$a = -\frac{1}{2}$$

The slope of $h(x)$ at $x = -1/2$ is $4/5$, and the slope of $h(x)$ at $x = 1/2$ is $4/5$.

■ 4. Find the value of a such that the tangent lines to $f(x) = 4x^3 - 6x + 7$ at $x = a$ and $x = a + 1$ are parallel.

© Krista King Math

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$f(a) = 4a^3 - 6a + 7$$

$$f'(x) = 12x^2 - 6$$

$$f'(a) = 12a^2 - 6$$

So the equation of the tangent line at $x = a$ is

$$y = f(a) + f'(a)(x - a)$$

$$y = 4a^3 - 6a + 7 + (12a^2 - 6)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$f(a + 1) = 4(a + 1)^3 - 6(a + 1) + 7$$

$$f(a + 1) = 4a^3 + 12a^2 + 6a + 5$$

$$f'(x) = 12x^2 - 6$$

$$f'(a + 1) = 12a^2 + 24a + 6$$

So the equation of the tangent line at $x = a + 1$ is

$$y = f(a + 1) + f'(a + 1)(x - (a + 1))$$

$$y = 4a^3 + 12a^2 + 6a + 5 + (12a^2 + 24a + 6)(x - (a + 1))$$

© Krista King Math

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$12a^2 - 6 = 12a^2 + 24a + 6$$

$$-12 = 24a$$

$$a = -\frac{1}{2}$$

The slope of $f(x)$ at $x = -1/2$ is 3, and the slope of $f(x)$ at $x = 1/2$ is 3.

■ 5. Find the value of a such that the tangent lines to $g(x) = (x-2)^3 + x^2 + 3$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$g(a) = (a-2)^3 + a^2 + 3$$

$$g'(x) = 3(x-2)^2 + 2x$$

$$g'(a) = 3(a-2)^2 + 2a$$

So the equation of the tangent line at $x = a$ is

$$y = g(a) + g'(a)(x - a)$$

© Krista King Math

$$y = (a-2)^3 + a^2 + 3 + (3(a-2)^2 + 2a)(x-a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$g(a+1) = (a+1-2)^3 + (a+1)^2 + 3$$

$$g(a+1) = (a-1)^3 + (a+1)^2 + 3$$

$$g'(a) = 3(a-2)^2 + 2a$$

$$g'(a+1) = 3(a+1-2)^2 + 2(a+1)$$

$$g'(a+1) = 3(a-1)^2 + 2(a+1)$$

So the equation of the tangent line at $x = a + 1$ is

$$y = g(a+1) + g'(a+1)(x-(a+1))$$

$$y = (a-1)^3 + (a+1)^2 + 3 + (3(a-1)^2 + 2(a+1))(x-(a+1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$3(a-2)^2 + 2a = 3(a-1)^2 + 2(a+1)$$

$$3a^2 - 10a + 12 = 3a^2 - 4a + 5$$

$$7 = 6a$$

$$a = \frac{7}{6}$$

The slope of $g(x)$ at $x = 7/6$ is $53/12$, and the slope of $g(x)$ at $x = 13/6$ is $53/12$.

© Krista King Math

■ 6. Find the approximate value of a, rounded to the nearest hundredth, such that the tangent lines to $h(x) = e^x - 3x^2$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$h(a) = e^a - 3a^2$$

$$h'(x) = e^x - 6x$$

$$h'(a) = e^a - 6a$$

So the equation of the tangent line at $x = a$ is

$$y = h(a) + h'(a)(x - a)$$

$$y = e^a - 3a^2 + (e^a - 6a)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$h(a + 1) = e^{a+1} - 3(a + 1)^2$$

$$h'(a) = e^a - 6a$$

$$h'(a + 1) = e^{a+1} - 6(a + 1)$$

So the equation of the tangent line at $x = a + 1$ is

$$y = h(a + 1) + h'(a + 1)(x - (a + 1))$$

© Krista King Math

$$y = e^{a+1} - 3(a+1)^2 + e^{a+1} - 6(a+1)(x-(a+1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$e^a - 6a = e^{a+1} - 6(a+1)$$

$$e^a - 6a = e^{a+1} - 6a - 6$$

$$e^a = e^{a+1} - 6$$

$$e^{a+1} - e^a - 6 = 0$$

$$e^a e^1 + (-e^a) - 6 = 0$$

$$(-e^a)e^1 - (-e^a) + 6 = 0$$

Substitute $x = -e^a$, then solve for x.

$$xe - x + 6 = 0$$

$$x(e-1) + 6 = 0$$

$$x(e-1) = -6$$

$$x = -\frac{6}{e-1}$$

Back-substitute.

$$-e^a = -\frac{6}{e-1}$$

$$e^a = \frac{6}{e-1}$$

© Krista King Math

Take the natural log of both sides.

$$\ln(e^a) = \ln\left(\frac{6}{e-1}\right)$$

$$a = \ln\left(\frac{6}{e-1}\right)$$

$$a \approx 1.25$$

The slope of $h(x)$ at $x = 1.25$ is -4, and the slope of $h(x)$ at $x = 2.25$ is -4.

© Krista King Math

VALUES THAT MAKE THE FUNCTION DIFFERENTIABLE

■ 1. What value of a and b will make the function differentiable?

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 3 \\ ax - b & \text{if} \quad x > 3 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x = 3$,

$$x^2 = ax - b$$

$$3^2 = a(3) - b$$

$$9 = 3a - b$$

$$b = 3a - 9$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x = 3$ must be equal to each other. So $2x = a$, and when $x = 3$, and $a = 6$. Therefore, $a = 6$ and

$$b = 3(6) - 9$$

$$b = 9$$

■ 2. What value of a and b will make the function differentiable?

© Krista King Math

$$g(x) = \begin{cases} ax + b & \text{if} \quad x \leq -1 \\ bx^2 - 1 & \text{if} \quad x > -1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $g(x)$ continuous at $x = -1$,

$$ax + b = bx^2 - 1$$

$$a(-1) + b = b(-1)^2 - 1$$

$$-a + b = b - 1$$

$$a = 1$$

If $g(x)$ is differentiable, then the derivatives of $g(x)$ at $x = -1$ must be equal to each other. So $a = 2bx$ when $x = -1$, and $a = 1$.

$$1 = 2b(-1)$$

$$b = -\frac{1}{2}$$

Therefore, $a = 1$ and $b = -1/2$.

■ 3. What value of a and b will make the function differentiable?

$$h(x) = \begin{cases} ax^3 & \text{if} \quad x \leq 2 \\ x^2 - b & \text{if} \quad x > 2 \end{cases}$$

© Krista King Math

Solution:

To be differentiable, the function has to be continuous. To make $h(x)$ continuous at $x = 2$,

$$ax^3 = x^2 - b$$

$$a(2)^3 = (2)^2 - b$$

$$8a = 4 - b$$

$$8a - 4 = -b$$

$$b = 4 - 8a$$

If $h(x)$ is differentiable, then the derivatives of $h(x)$ at $x = 2$ must be equal to each other. So $3ax^2 = 2x$ when $x = 2$, and

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

To get b, we'll plug in $a = 1/3$.

$$b = 4 - 9\left(\frac{1}{3}\right) = \frac{4}{3}$$

Therefore, $a = 1/3$ and $b = 4/3$.

© Krista King Math

■ 4. What value of a and b will make the function differentiable?

$$f(x) = \begin{cases} 3 - x & \text{if} \quad x \leq 1 \\ ax^2 - bx & \text{if} \quad x > 1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x = 1$,

$$3 - x = ax^2 - bx$$

$$3 - (1) = a(1)^2 - b(1)$$

$$2 = a - b$$

$$a = 2 + b$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x = 1$ must be equal to each other. So

$$-1 = 2ax - b$$

$$-1 = 2a(1) - b$$

$$-1 = 2a - b$$

$$-1 - 2a = -b$$

$$b = 2a + 1$$

© Krista King Math

Now, since $a = 2 + b$ and $b = 2a + 1$,

$$a = 2 + 2a + 1$$

$$-a = 3$$

$$a = -3$$

Then

$$b = 2a + 1$$

$$b = 2(-3) + 1$$

$$b = -5$$

The answer is $a = -3$ and $b = -5$.

■ 5. What value of a and b will make the function differentiable?

$$g(x) = \begin{cases} x^3 & \text{if} \quad x \leq 1 \\ a(x-2)^2 - b & \text{if} \quad x > 1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $g(x)$ continuous at $x = 1$,

$$x^3 = a(x-2)^2 - b$$

© Krista King Math

$$(1)^3 = a(1-2)^2 - b$$

$$1 = a - b$$

$$-b = 1 - a$$

$$b = a - 1$$

If $g(x)$ is differentiable, then the derivatives of $g(x)$ at $x = 1$ must be equal to each other. So

$$3x^2 = 2a(x-2)$$

$$3(1) = 2a(1-2)$$

$$3 = 2a(-1)$$

$$a = -\frac{3}{2}$$

So $a = -3/2$ and $b = -(3/2) - 1 = -(5/2)$.

6. What value of a and b will make the function differentiable?

$$h(x) = \begin{cases} ax^2 + b & \text{if} \quad x \leq 3 \\ bx + 4 & \text{if} \quad x > 3 \end{cases}$$

Solution:

© Krista King Math

To be differentiable, the function has to be continuous. To make $h(x)$ continuous at $x = 3$,

$$ax^2 + b = bx + 4$$

$$a(3)^2 + b = b(3) + 4$$

$$9a + b = 3b + 4$$

$$9a = 2b + 4$$

$$a = \frac{2b + 4}{9}$$

If $h(x)$ is differentiable, then the derivatives of $h(x)$ at $x = 3$ must be equal to each other. So

$$2ax = b$$

$$2a(3) = b$$

$$b = 6a$$

Plugging $b = 6a$ into the equation for a gives

$$a = \frac{2(6a) + 4}{9}$$

$$9a = 12a + 4$$

$$-3a = 4$$

$$a = -\frac{4}{3}$$

© Krista King Math

Then, $b = 6(-4/3) = -8$. The answer is $a = -4/3$ and $b = -8$.

■ 7. What value of a and b will make the function differentiable?

$$f(x) = \begin{cases} ax^2 + 10 & \text{if} \quad x \leq 2 \\ x^2 - 6x + b & \text{if} \quad x > 2 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x = 2$,

$$ax^2 + 10 = x^2 - 6x + b$$

$$a(2)^2 + 10 = (2)^2 - 6(2) + b$$

$$4a + 10 = 4 - 12 + b$$

$$b = 4a + 18$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x = 2$ must be equal to each other. So

$$2ax = 2x - 6$$

$$2a(2) = 2(2) - 6$$

$$4a = -2$$

© Krista King Math

$$a = -\frac{1}{2}$$

Plugging $a = -1/2$ into the equation for b gives

$$b = 4\left(-\frac{1}{2}\right) + 18$$

$$b = -2 + 18$$

$$b = 16$$

The answer is $a = -1/2$ and $b = 16$.

© Krista King Math

HORIZONTAL AND VERTICAL TANGENT LINES AND DIFFERENTIABILITY

■ 1. Find the equation of the vertical tangent line in the interval $[2,6]$.

$$f(x) = 2\sqrt[3]{x-4} + 2$$

Solution:

Find the derivative.

$$f(x) = 2(x-4)^{\frac{1}{3}} + 2$$

$$f'(x) = 2\left(\frac{1}{3}\right)(x-4)^{-\frac{2}{3}}$$

$$f'(x) = \frac{2}{3\sqrt[3]{(x-4)^2}}$$

Identify the x-value that makes the derivative undefined. The denominator

$$3\sqrt[3]{(x-4)^2} = 0$$

is 0 when $x = 4$. Next, find $f(4)$, which is

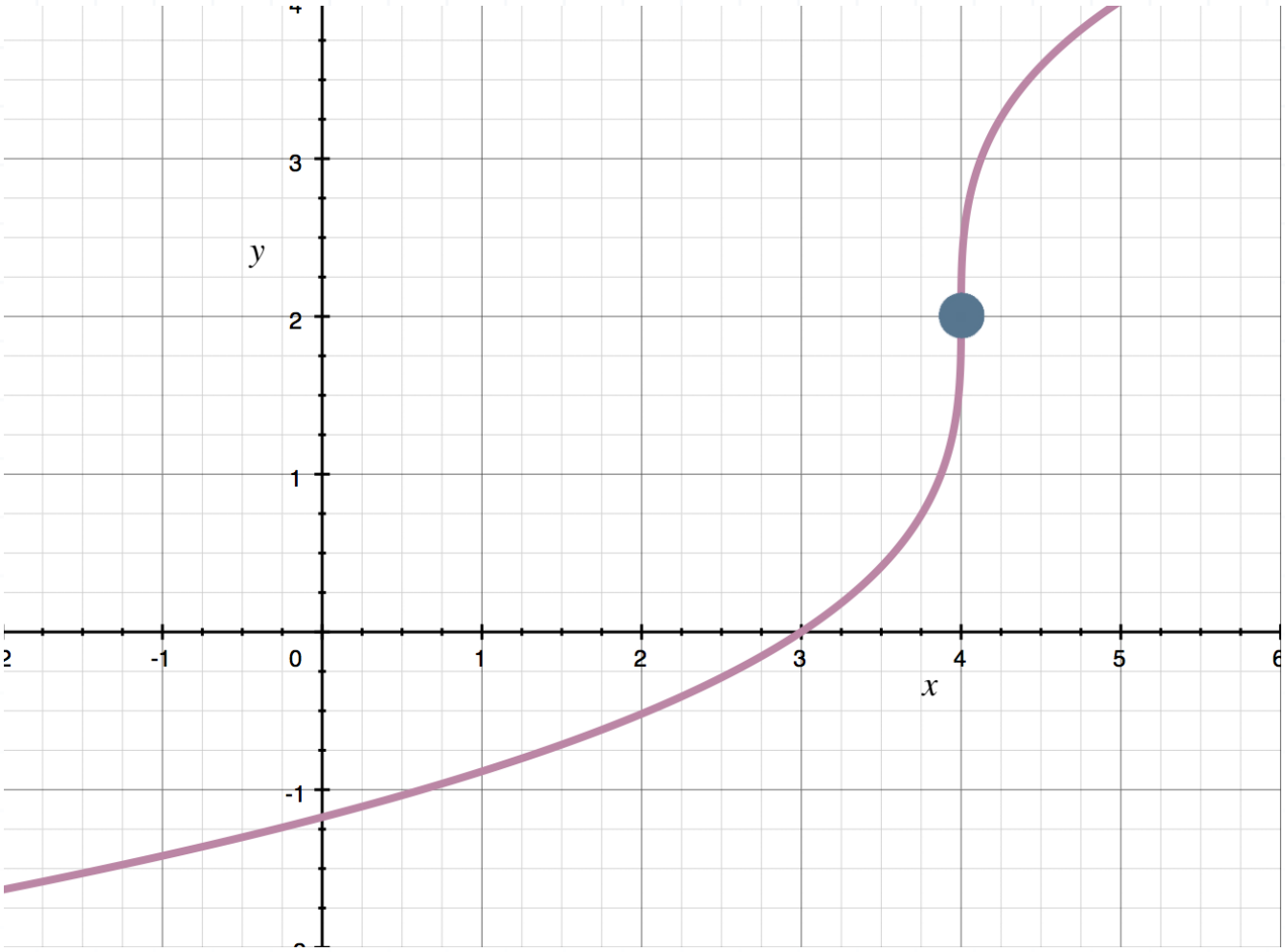
$$f(4) = 2 \cdot \sqrt[3]{4-4} + 2$$

$$f(4) = 2(0) + 2$$

$$f(4) = 2$$

© Krista King Math

The tangent is a vertical line that passes through the point $(4,2)$. Therefore, the vertical tangent is $x = 4$. The graph below shows the point of tangency.



■ 2. Find the equation of the horizontal tangent line in the interval $[2,4]$.

$$g(x) = \frac{3}{2}(x-2)^3 - 2x + 3$$

Solution:

Find the derivative.

$$g(x) = \frac{3}{2}(x-2)^3 - 2x + 3$$

© Krista King Math

$$g'(x) = 3\left(\frac{3}{2}\right)(x-2)^2 - 2$$

$$g'(x) = \frac{9}{2}(x-2)^2 - 2$$

There's no x-value that makes the derivative undefined, so there will be no vertical tangent line. But there will be a horizontal tangent line where the derivative is equal to 0.

$$\frac{9}{2}(x-2)^2 - 2 = 0$$

$$\frac{9}{2}(x-2)^2 = 2$$

$$(x-2)^2 = \frac{4}{9}$$

$$x - 2 = \pm\frac{2}{3}$$

$$x = 2 \pm \frac{2}{3}$$

$$x = \frac{4}{3}, \frac{8}{3}$$

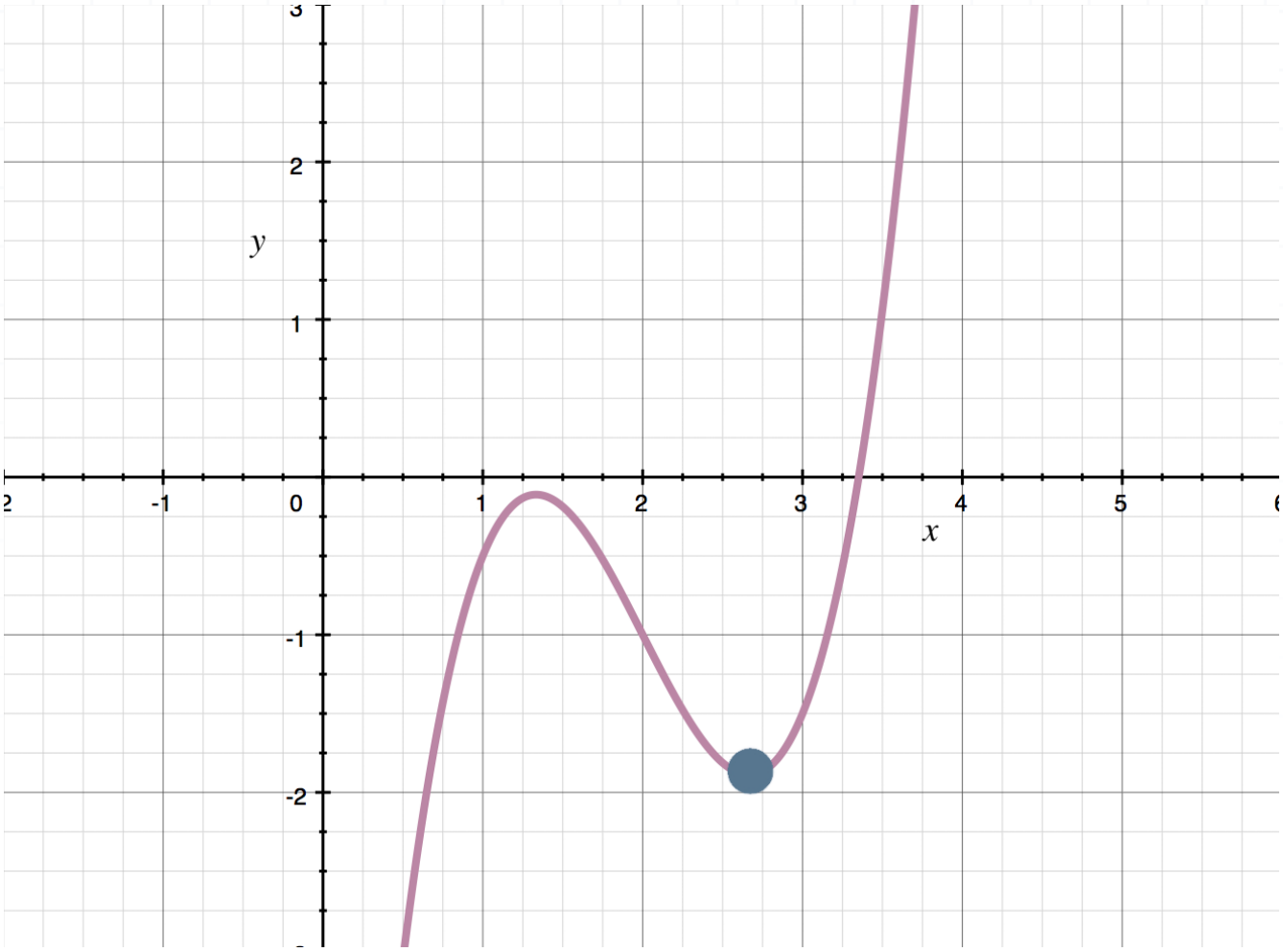
We can ignore the value $x = 4/3$, since it's outside the interval $[2,4]$ that we were asked about. Evaluate the function at $x = 8/3$,

$$g\left(\frac{8}{3}\right) = \frac{3}{2}\left(\frac{8}{3} - 2\right)^3 - 2\left(\frac{8}{3}\right) + 3$$

© Krista King Math

$$g\left(\frac{8}{3}\right) = -\frac{17}{9}$$

The tangent is a horizontal line that passes through the point $(8/3, -17/9)$. Therefore, the horizontal tangent is $y = -17/9$. The graph below shows the point of tangency.



■ 3. Find the equation of the horizontal and vertical tangent lines in the interval $[-2,2]$.

$$h(x) = \frac{x^3}{3 - x^2} + 1$$

© Krista King Math

Solution:

Find the derivative.

$$h(x) = \frac{x^3}{3 - x^2} + 1$$

$$h'(x) = \frac{(3 - x^2)(3x^2) - (x^3)(-2x)}{(3 - x^2)^2}$$

$$h'(x) = \frac{-x^4 + 9x^2}{(3 - x^2)^2}$$

$$h'(x) = -\frac{x^2(x^2 - 9)}{(3 - x^2)^2}$$

Identify the x-value that makes the derivative undefined. The denominator

$$(3 - x^2)^2 = 0$$

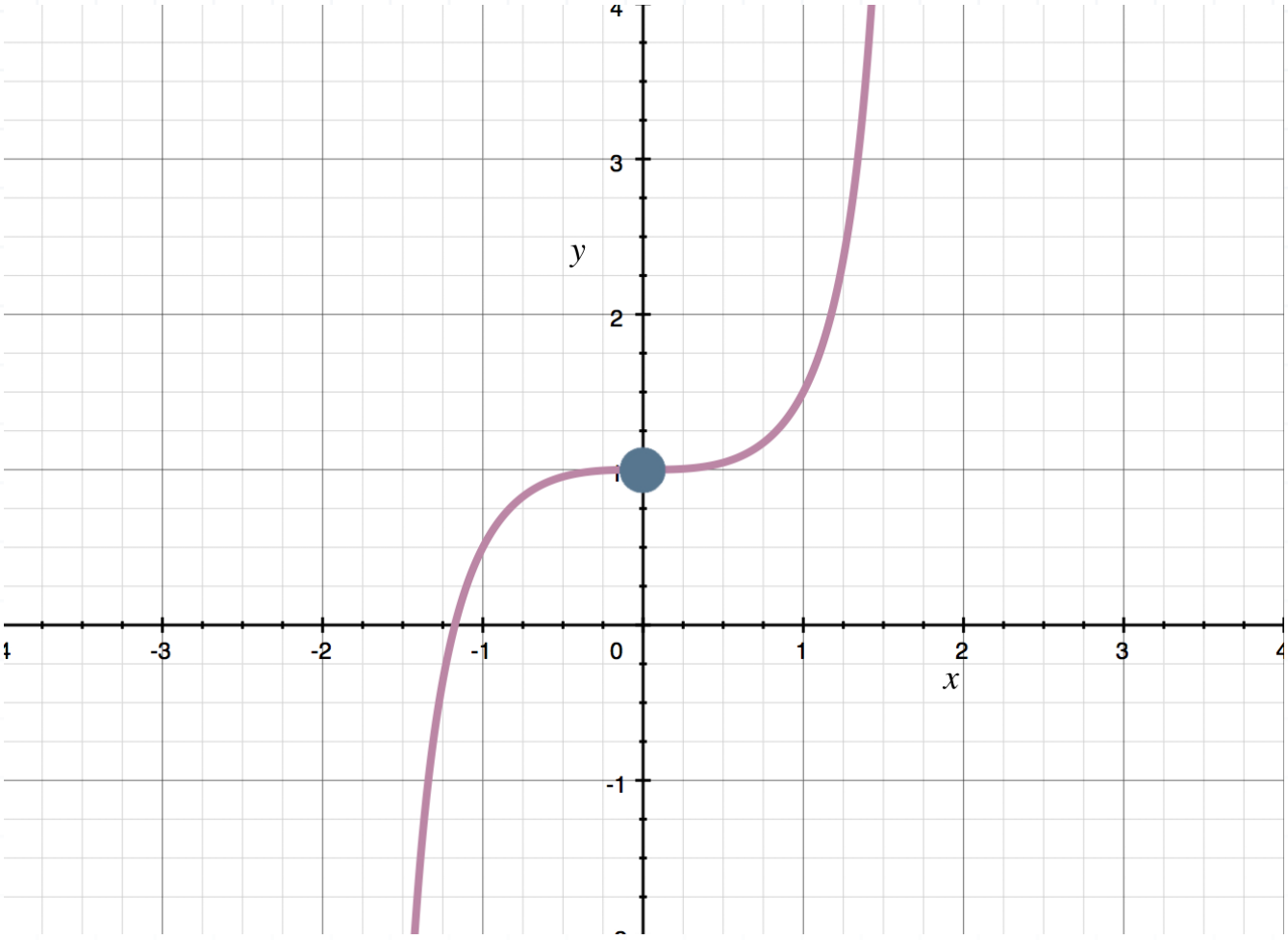
is 0 when $x = \pm\sqrt{3}$. The derivative will be equal to 0 when the numerator is equal to 0.

$$x^2(x^2 - 9) = 0$$

$$x = -3,\ 0,\ 3$$

The only one of those values in $[-2,2]$ is $x = 0$. The tangent line at $x = 0$ is horizontal and passes through $(0,1)$. Therefore, the horizontal tangent is $y = 1$. The graph below shows the point of tangency for the horizontal tangent line, and the vertical tangent lines at $x = \pm\sqrt{3}$ bound the middle section of the graph.

© Krista King Math



■ 4. Find the equation of the horizontal tangent line in the interval $[-2,3]$.

$f(x) = 3x^2 - 6x - 4$

Solution:

Find the derivative.

$$f(x) = 3x^2 - 6x - 4$$

$$f'(x) = 6x - 6$$

© Krista King Math

There's no x-value that makes the derivative undefined, so there will be no vertical tangent line. But there will be a horizontal tangent line where the derivative is equal to 0.

$$6x - 6 = 0$$

$$6x = 6$$

$$x = 1$$

Evaluate the function at $x = 1$.

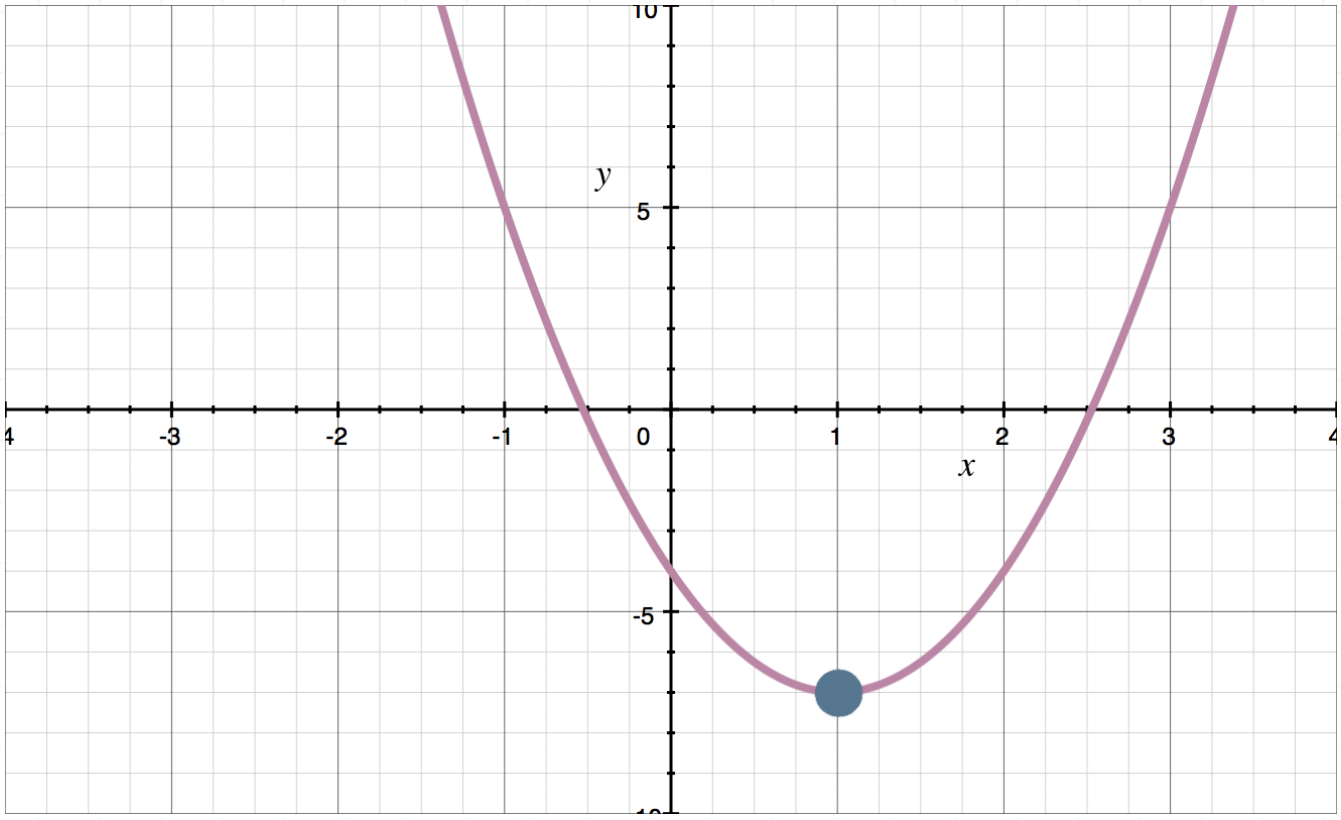
$$f(1) = 3(1)^2 - 6(1) - 4$$

$$f(1) = 3 - 6 - 4$$

$$f(1) = -7$$

The tangent is a horizontal line that passes through the point $(1, -7)$. Therefore, the horizontal tangent is $y = -7$. The graph below shows the point of tangency.

© Krista King Math



5. Find the equation of the vertical tangent line in the interval $[3,6]$.

$$g(x) = \sqrt{3(x-3)^2 - 3}$$

Solution:

Find the derivative.

$$g(x) = \sqrt{3(x-3)^2 - 3}$$

$$g(x) = \sqrt{3x^2 - 18x + 24}$$

$$g'(x) = \frac{6x - 18}{2\sqrt{3x^2 - 18x + 24}}$$

© Krista King Math

$$g'(x) = \frac{3x - 9}{\sqrt{3x^2 - 18x + 24}}$$

Identify the x-value that makes the derivative undefined. The denominator

$$\sqrt{3x^2 - 18x + 24} = 0$$

$$3x^2 - 18x + 24 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x - 4)(x - 2) = 0$$

$$x = 2,\ 4$$

is 0 when $x = 2,\ 4$. We only need to consider $x = 4$, since that's the only value in the interval $[3,6]$. Next, find $g(4)$.

$$g(4) = \sqrt{3(4 - 3)^2 - 3}$$

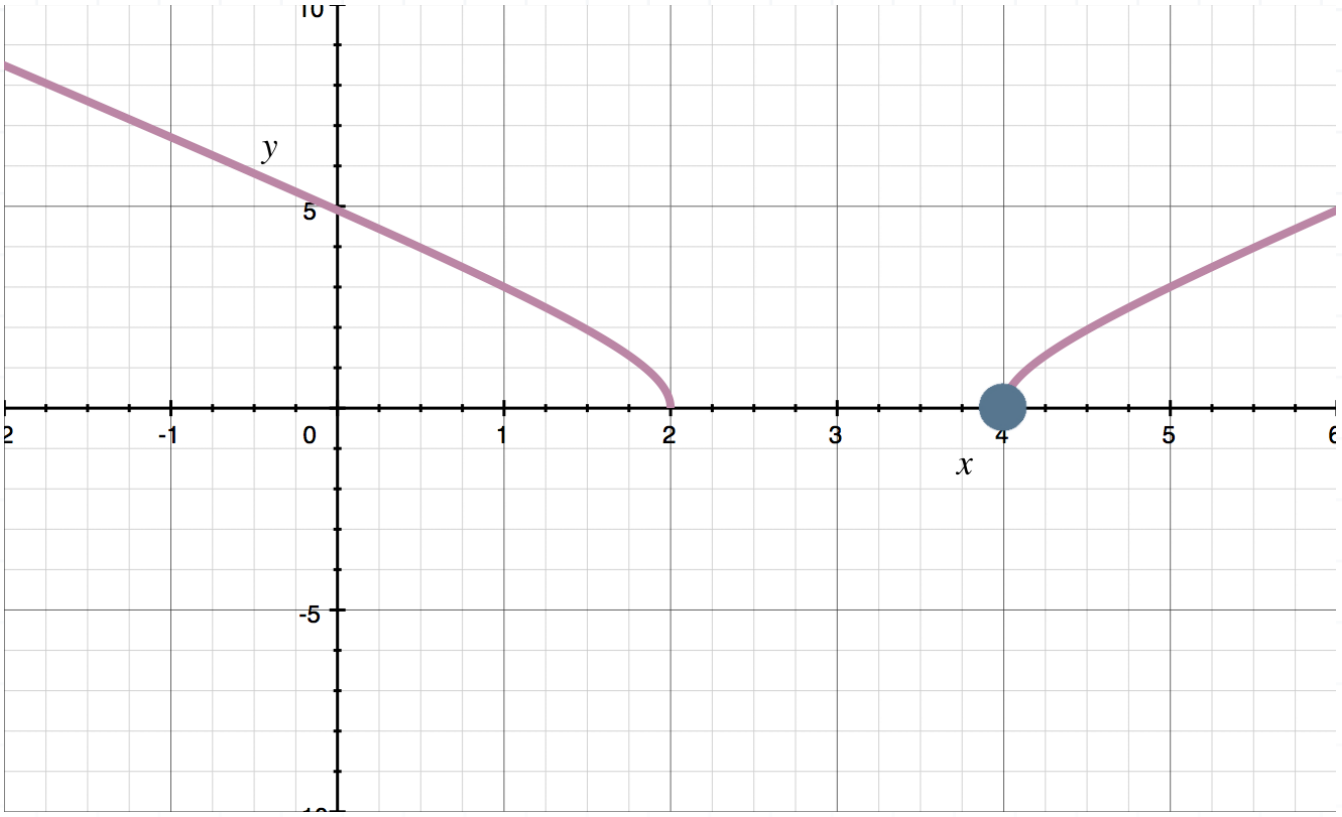
$$g(4) = \sqrt{3(1)^2 - 3}$$

$$g(4) = \sqrt{0}$$

$$g(4) = 0$$

The tangent is a vertical line that passes through $(4,0)$. Therefore, the vertical tangent is $x = 4$. The graph below shows the point of tangency.

© Krista King Math



© Krista King Math

NORMAL LINES

■ 1. Find the equation of the normal line to the graph of $f(x) = 5x^4 + 3e^x$ at $(0,3)$.

Solution:

Begin by finding the tangent line at $(0,3)$, starting with taking the derivative. Then evaluate the derivative at $(0,3)$.

$$f'(x) = 20x^3 + 3e^x$$

$$f'(0) = 20(0)^3 + 3e^0$$

$$f'(0) = 0 + 3(1)$$

$$f'(0) = 3$$

With $f'(0) = 3$ and $(a, f(a)) = (0,3)$, the tangent line is

$$y = f(a) + f'(a)(x - a)$$

$$y = 3(x - 0) + 3$$

$$y = 3x + 3$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/3$, so the equation of the normal line is

© Krista King Math

$$y = 3 - \frac{1}{3}(x - 0)$$

$$y = -\frac{1}{3}x + 3$$

■ 2. Find the equation of the normal line to the graph of $g(x) = \ln e^{4x} + 2x^3$ at $(2,24)$.

Solution:

Begin by finding the tangent line at $(2,24)$, starting with taking the derivative. Then evaluate the derivative at $(2,24)$.

$$g'(x) = 4 + 6x^2$$

$$g'(2) = 4 + 6(2)^2$$

$$g'(2) = 4 + 24$$

$$g'(2) = 28$$

With $g'(2) = 28$ and $(a, g(a)) = (2,24)$, the tangent line is

$$y = g(a) + g'(a)(x - a)$$

$$y = 24 + 28(x - 2)$$

$$y = 28x - 32$$

© Krista King Math

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/28$, so the equation of the normal line is

$$y = 24 - \frac{1}{28}(x - 2)$$

$$y = -\frac{1}{28}x + \frac{337}{14}$$

■ 3. Find the equation of the normal line to the graph of $h(x) = 5\cos x + 5\sin x$ at $(\pi/2,5)$.

Solution:

Begin by finding the tangent line at $(\pi/2,5)$, starting with taking the derivative. Then evaluate the derivative at $(\pi/2,5)$.

$$h'(x) = -5\sin x + 5\cos x$$

$$h'\left(\frac{\pi}{2}\right) = -5\sin\left(\frac{\pi}{2}\right) + 5\cos\left(\frac{\pi}{2}\right)$$

$$h'\left(\frac{\pi}{2}\right) = -5(1) + 5(0)$$

$$h'\left(\frac{\pi}{2}\right) = -5$$

With $h'(\pi/2) = -5$ and $(a, h(a)) = (\pi/2,5)$, the tangent line is

© Krista King Math

$$y = h(a) + h'(a)(x - a)$$

$$y = 5 - 5\left(x - \frac{\pi}{2}\right)$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $1/5$, so the equation of the normal line is

$$y = 5 + \frac{1}{5}\left(x - \frac{\pi}{2}\right)$$

■ 4. Find the equation of the normal line to the graph of $f(x) = 7x^3 + 2x^2 - 5x + 9$ at $(2,63)$.

Solution:

Begin by finding the tangent line at $(2,63)$, starting with taking the derivative. Then evaluate the derivative at $(2,63)$.

$$f'(x) = 21x^2 + 4x - 5$$

$$f'(2) = 21(2)^2 + 4(2) - 5$$

$$f'(2) = 84 + 8 - 5$$

$$f'(2) = 87$$

With $f'(2) = 87$ and $(a, f(a)) = (2,63)$, the tangent line is

© Krista King Math

$$y = f(a) + f'(a)(x - a)$$

$$y = 63 + 87(x - 2)$$

$$y = 87x - 111$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $1/87$, so the equation of the normal line is

$$y = 63 - \frac{1}{87}(x - 2)$$

$$y = -\frac{1}{87}x + \frac{5{,}483}{87}$$

■ 5. Find the equation of the normal line to the graph of $g(x) = 5\sqrt{x^2 - 14x + 49}$ at $(2{,}25)$.

Solution:

Begin by finding the tangent line at $(2{,}25)$, starting with taking the derivative. Then evaluate the derivative at $(2{,}25)$.

$$g'(x) = \frac{5}{2\sqrt{x^2 - 14x + 49}} \cdot (2x - 14)$$

$$g'(x) = \frac{5x - 35}{\sqrt{x^2 - 14x + 49}}$$

© Krista King Math

$$g'(x) = \frac{5(x-7)}{|x-7|}$$

$$g'(2) = \frac{5(2-7)}{|2-7|}$$

$$g'(2) = \frac{-25}{5}$$

$$g'(2) = -5$$

With $g'(2) = -5$ and $(a, g(a)) = (2,25)$, the tangent line is

$$y = g(a) + g'(a)(x-a)$$

$$y = 25 - 5(x-2)$$

$$y = -5x + 35$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $1/5$, so the equation of the normal line is

$$y = 25 + \frac{1}{5}(x-2)$$

$$y = \frac{1}{5}x + \frac{123}{5}$$

© Krista King Math

AVERAGE RATE OF CHANGE

■ 1. Find the average rate of change of the function over the interval $[4,9]$.

$$f(x) = \frac{5\sqrt{x} - 2}{3}$$

Solution:

In this question, $f(9)$ and $f(4)$ are

$$f(9) = \frac{5\sqrt{9} - 2}{3} = \frac{5(3) - 2}{3} = \frac{13}{3}$$

$$f(4) = \frac{5\sqrt{4} - 2}{3} = \frac{5(2) - 2}{3} = \frac{8}{3}$$

Therefore, average rate of change on $[a, b] = [4,9]$ is given by

$$\frac{f(b) - f(a)}{b - a}$$

$$\frac{\frac{13}{3} - \frac{8}{3}}{9 - 4} = \frac{\frac{5}{3}}{\frac{5}{1}} = \frac{5}{3} \cdot \frac{1}{5} = \frac{1}{3}$$

■ 2. Find the average rate of change of the function over the interval $[16,25]$.

© Krista King Math

$$g(x) = \frac{2x - 8}{\sqrt{x} - 2}$$

Solution:

In this question, $g(25)$ and $g(16)$ are

$$g(25) = \frac{2(25) - 8}{\sqrt{25} - 2} = \frac{42}{3} = 14$$

$$g(16) = \frac{2(16) - 8}{\sqrt{16} - 2} = \frac{24}{2} = 12$$

Therefore, average rate of change on $[a, b] = [16,25]$ is given by

$$\frac{g(b) - g(a)}{b - a}$$

$$\frac{14 - 12}{25 - 16} = \frac{2}{9}$$

■ 3. Find the average rate of change of the function over the interval $[0,4]$.

$$h(x) = \frac{x^3 - 8}{x^2 - 4x - 5}$$

Solution:

© Krista King Math

In this question, $g(4)$ and $g(0)$ are

$$h(4) = \frac{4^3 - 8}{4^2 - 4(4) - 5} = \frac{64 - 8}{16 - 16 - 5} = \frac{56}{-5} = -\frac{56}{5}$$

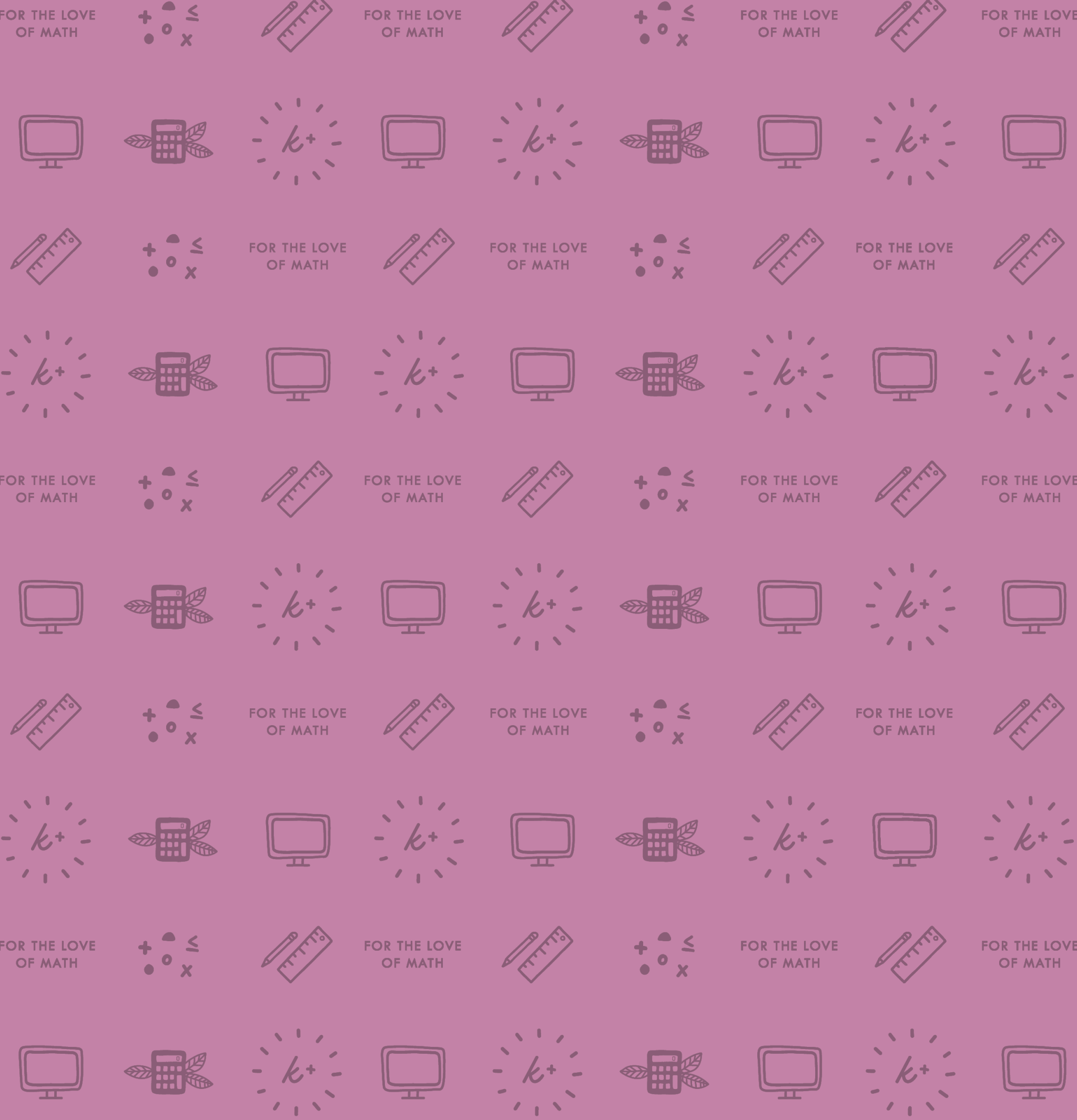
$$h(0) = \frac{0^3 - 8}{0^2 - 4(0) - 5} = \frac{0 - 8}{0 - 0 - 5} = \frac{-8}{-5} = \frac{8}{5}$$

Therefore, average rate of change on $[a, b] = [0,4]$ is given by

$$\frac{h(b) - h(a)}{b - a}$$

$$\frac{\frac{-56}{5} - \frac{8}{5}}{4 - 0} = \frac{-\frac{64}{5}}{\frac{4}{1}} = -\frac{64}{5} \cdot \frac{1}{4} = -\frac{16}{5}$$

© Krista King Math



WWW.KRISTAKINGMATH.COM